

Integer Powers of Certain Complex Pentadiagonal 2–Toeplitz Matrices

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Abstract

In this study, we get a general expression for the entries of the s th power of even order pentadiagonal 2-Toeplitz matrices.

1 Introduction

Gover [1] gained eigenvalues and eigenvectors of a tridiagonal 2-Toeplitz matrix in terms of the Chebyshev polynomials. Hadj and Elouafi [2] obtained the general expression of the characteristic polynomial, determinant and eigenvectors for pentadiagonal matrices. Rimas [3] offered general expression for the entries of the power of tridiagonal 2-Toeplitz matrix, in terms of the Chebyshev polynomials of the second kind. Obvious formulas for the determinants of a band symmetric Toeplitz matrix is restated in [4]. Álvarez-Nodarse et al. [5] gained the general expressions for the eigenvalues, eigenvectors and the spectral measure of 2 and 3-Toeplitz matrices. The powers of even order symmetric pentadiagonal matrices are calculated in [6]. Öteleş and Akbulak [7] viewed powers of tridiagonal matrices. Wu [8] calculated the powers of Toeplitz Matrices. The powers of complex pentadiagonal Toeplitz matrices are computed in [9].

This paper is organized as follows: the first section, motivated by [2], we apply K_n pentadiagonal 2-Toeplitz matrix to the characteristic polynomial and eigenvectors of this matrix in given [2]. In Section 2, we obtain the eigenvalues and eigenvectors of K_n pentadiagonal 2-Toeplitz matrix. In Section 3, the s th power of pentadiagonal 2-Toeplitz matrix we will get by using the expression $K_n^s = L_n J_n^s L_n^{-1}$ [11], where J_n is the Jordan's form of K_n and L_n is the transforming matrix. In Section 4, some numerical examples are given.

Consider the polynomial sequence $\{A_i\}_{i \geq 0}$ and $\{B_i\}_{i \geq 0}$ characterized by a three-term recurrence relation

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$$\begin{aligned}
xA_0(x) &= a_1A_0(x) + b_1A_2(x) \\
xA_1(x) &= a_2A_1(x) + b_2A_3(x) \\
xA_{i-1}(x) &= c_1A_{i-3}(x) + a_1A_{i-1}(x) + b_1A_{i+1}(x) \text{ for } i \geq 3 \text{ and } i = 2t+1 (t \in \mathbb{N}) \\
xA_{i-1}(x) &= c_2A_{i-3}(x) + a_2A_{i-1}(x) + b_2A_{i+1}(x) \text{ for } i \geq 3 \text{ and } i = 2t (t \in \mathbb{N})
\end{aligned} \tag{1}$$

with initial conditions $A_0(x) = 0$ and $A_1(x) = 1$, and

$$\begin{aligned}
xB_0(x) &= a_1B_0(x) + b_1B_2(x) \\
xB_1(x) &= a_2B_1(x) + b_2B_3(x) \\
xB_{i-1}(x) &= c_1B_{i-3}(x) + a_1B_{i-1}(x) + b_1B_{i+1}(x) \text{ for } i \geq 3 \text{ and } i = 2t+1 (t \in \mathbb{N}) \\
xB_{i-1}(x) &= c_2B_{i-3}(x) + a_2B_{i-1}(x) + b_2B_{i+1}(x) \text{ for } i \geq 3 \text{ and } i = 2t (t \in \mathbb{N})
\end{aligned} \tag{2}$$

with initial conditions $B_0(x) = 1$ and $B_1(x) = 0$, here $a_1, a_2 \in \mathbb{C}$ and $b_1, b_2, c_1, c_2 \in \mathbb{C} \setminus \{0\}$. We can write a matrix form to this three-term recurrence relations

$$\begin{aligned}
xA_{n-1}(x) &= K_n A_{n-1}(x) + A_n(x)d_{n-1} + A_{n+1}(x)d_n \\
xB_{n-1}(x) &= K_n B_{n-1}(x) + B_n(x)d_{n-1} + B_{n+1}(x)d_n
\end{aligned} \tag{3}$$

where $A_{n-1}(x) = [A_0(x), A_1(x), A_2(x), \dots, A_{n-1}(x)]^T$,
 $B_{n-1}(x) = [B_0(x), B_1(x), B_2(x), \dots, B_{n-1}(x)]^T$,

$$\left. \begin{aligned} d_{n-1} &= [0, 0, 0, \dots, 0, b_2, 0]^T \\ d_n &= [0, 0, 0, \dots, 0, 0, b_1]^T \end{aligned} \right\} (n = 2t+1, t \in \mathbb{N})$$

and

$$\left. \begin{aligned} d_{n-1} &= [0, 0, 0, \dots, 0, b_1, 0]^T \\ d_n &= [0, 0, 0, \dots, 0, 0, b_2]^T \end{aligned} \right\} (n = 2t, t \in \mathbb{N}).$$

Let

$$K_n = \begin{bmatrix} a_1 & 0 & b_1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & a_2 & 0 & b_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ c_1 & 0 & a_1 & 0 & b_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & c_2 & 0 & a_2 & 0 & b_2 & \cdots & 0 & 0 & 0 \\ 0 & 0 & c_1 & 0 & a_1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & b_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & a_1 & 0 & b_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & a_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & c_1 & 0 & a_1 \end{bmatrix} \text{ for } n = 2t+1 (t \in \mathbb{N}) \tag{4}$$

and

$$K_n = \begin{bmatrix} a_1 & 0 & b_1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & a_2 & 0 & b_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ c_1 & 0 & a_1 & 0 & b_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & c_2 & 0 & a_2 & 0 & b_2 & \cdots & 0 & 0 & 0 \\ 0 & 0 & c_1 & 0 & a_1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & b_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & a_2 & 0 & b_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & a_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & c_2 & 0 & a_2 \end{bmatrix} \text{ for } n = 2t \ (t \in \mathbb{N}) \quad (5)$$

here $a_1, a_2 \in \mathbb{C}$ and $b_1, b_2, c_1, c_2 \in \mathbb{C} \setminus \{0\}$.

Lemma 1 *The polynomial sequences $\{A_i\}_{i \geq 0}$ and $\{B_i\}_{i \geq 0}$ confirm*

$$\begin{aligned} \deg(A_{2i+1}) &= i \text{ and the leading coefficient of } A_{2i+1} \text{ is equal to } \frac{1}{b_2^i}, \\ \deg(A_{2i}) &= 0, \\ \deg(B_{2i+1}) &= 0, \\ \deg(B_{2i}) &= i \text{ and the leading coefficient of } B_{2i} \text{ is equal to } \frac{1}{b_1^i}. \end{aligned}$$

Proof. Let us prove by the inductive method. For the basis step, we possess

For $i = 0$: $A_0(x) = 0, B_0(x) = 1$.

For $i = 1$: $A_1(x) = 1, B_1(x) = 0$.

For $i = 2$: $A_2(x) = 0, B_2(x) = \frac{x-a_1}{b_1}$.

For $i = 3$: $A_3(x) = \frac{x-a_2}{b_2}, B_3(x) = 0$.

For $i = 4$: $A_4(x) = 0, B_4(x) = \frac{(x-a_1)^2 - b_1 c_1}{b_1^2}$.

Assuming that (1) and (2) are correct for $p = k \geq 3$. We will prove it for $k = p + 1$, we obtain

$$\begin{aligned} xA_{p-1}(x) &= c_1 A_{p-3}(x) + a_1 A_{p-1}(x) + b_1 A_{p+1}(x) \text{ for } p \geq 3 \text{ and } p = 2t + 1 \ (t \in \mathbb{N}) \\ xA_{p-1}(x) &= c_2 A_{p-3}(x) + a_2 A_{p-1}(x) + b_2 A_{p+1}(x) \text{ for } p \geq 3 \text{ and } p = 2t \ (t \in \mathbb{N}) \\ xB_{p-1}(x) &= c_1 B_{p-3}(x) + a_1 B_{p-1}(x) + b_1 B_{p+1}(x) \text{ for } p \geq 3 \text{ and } p = 2t + 1 \ (t \in \mathbb{N}) \\ xB_{p-1}(x) &= c_2 B_{p-3}(x) + a_2 B_{p-1}(x) + b_2 B_{p+1}(x) \text{ for } p \geq 3 \text{ and } p = 2t \ (t \in \mathbb{N}). \end{aligned} \quad (6)$$

If $p := 2i$, then we have

$$\begin{aligned} xA_{2i-1}(x) &= c_2 A_{2i-3}(x) + a_2 A_{2i-1}(x) + b_2 A_{2i+1}(x) \\ xB_{2i-1}(x) &= c_2 B_{2i-3}(x) + a_2 B_{2i-1}(x) + b_2 B_{2i+1}(x). \end{aligned}$$

Accordingly

$$\begin{aligned} \deg(A_{2i+1}(x)) &= \deg(xA_{2i-1}(x)) \\ \deg(B_{2i+1}(x)) &= 0 \end{aligned}$$

and the leading coefficient of

$$\begin{aligned} A_{2i+1} &= \frac{1}{b_2^2} (\text{leading coefficient of } A_{2i-1}(x)), \\ &= \frac{1}{b_2} \frac{1}{(b_2)^{i-1}} = \frac{1}{(b_2)^i}. \end{aligned} \quad (7)$$

If $p := 2i - 1$, then we write

$$\begin{aligned} xA_{2i-2}(x) &= c_1 A_{2i-4}(x) + a_1 A_{2i-2}(x) + b_1 A_{2i}(x) \\ xB_{2i-2}(x) &= c_1 B_{2i-4}(x) + a_1 B_{2i-2}(x) + b_1 B_{2i}(x). \end{aligned}$$

So,

$$\begin{aligned} \deg(A_{2i}(x)) &= 0 \\ \deg(B_{2i}(x)) &= \deg(xB_{2i-2}(x)) \end{aligned}$$

and the leading coefficient of

$$\begin{aligned} B_{2i} &= \frac{1}{b_1} (\text{leading coefficient of } B_{2i-2}(x)), \\ &= \frac{1}{b_1} \frac{1}{(b_1)^{i-1}} = \frac{1}{(b_1)^i}. \end{aligned} \quad (8)$$

■

Definition 2 K_n be n -square pentadiagonal 2-Toeplitz matrix, one correlates the sequence polynomial P_i described by

$$P_i = \det \begin{bmatrix} A_n & A_i \\ B_n & B_i \end{bmatrix}. \quad (9)$$

Lemma 3 Due to (1) and (2), we own

$$\begin{aligned} xP_{n-1}(x) &= K_n P_{n-1}(x) + P_n(x)d_{n-1} + P_{n+1}(x)d_n \\ &= K_n P_{n-1}(x) + P_{n+1}(x)d_n \end{aligned} \quad (10)$$

where $P_{n-1}(x) = [P_0(x), P_1(x), P_2(x), \dots, P_{n-1}(x)]^T$,

$$\left. \begin{aligned} d_{n-1} &= [0, 0, 0, \dots, 0, b_2, 0]^T \\ d_n &= [0, 0, 0, \dots, 0, 0, b_1]^T \end{aligned} \right\} (n = 2t + 1, t \in \mathbb{N}),$$

$$\left. \begin{aligned} d_{n-1} &= [0, 0, 0, \dots, 0, b_1, 0]^T \\ d_n &= [0, 0, 0, \dots, 0, 0, b_2]^T \end{aligned} \right\} (n = 2t, t \in \mathbb{N}).$$

Lemma 4 The polynomial P_{n+1} is degree n and the leading coefficient of P_{n+1} is

$$\begin{aligned} &-\frac{1}{(b_1 b_2)^i}, \quad \text{if } n = 2t \ (t \in \mathbb{N}) \\ &\frac{1}{b_1^{i+1} b_2^i}, \quad \text{if } n = 2t + 1 \ (t \in \mathbb{N}). \end{aligned} \quad (11)$$

Proof. If $n = 2i$

$$P_{2i+1} = \det \begin{bmatrix} A_{2i} & A_{2i+1} \\ B_{2i} & B_{2i+1} \end{bmatrix} = A_{2i}B_{2i+1} - A_{2i+1}B_{2i}$$

and using Lemma 1

$$\deg(P_{2i+1}) = \deg(A_{2i+1}B_{2i})$$

and the leading coefficient of

$$P_{2i+1} = -\frac{1}{b_2^i} \frac{1}{b_1^i} = -\frac{1}{(b_1b_2)^i}.$$

If $n = 2i + 1$

$$P_{2i+1} = \det \begin{bmatrix} A_{2i+1} & A_{2i+2} \\ B_{2i+1} & B_{2i+2} \end{bmatrix} = A_{2i+1}B_{2i+2} - A_{2i+2}B_{2i+1}$$

and using Lemma 1

$$\deg(P_{2i+2}) = \deg(A_{2i+1}B_{2i+2})$$

and the leading coefficient of

$$P_{2i+2} = \frac{1}{b_2^i} \frac{1}{b_1^{i+1}}.$$

■

Lemma 5 *If α is a zero of the polynomial P_{n+1} then α is an eigenvalue of the matrix K_n .*

Proof. Let α is a zero of the polynomial P_{n+1} , from equation (10), we acquire

$$K_n P_{n-1}(\alpha) = \alpha P_{n-1}(\alpha).$$

There are four cases to be noted.

Case I. Suppose for $n = 2t$ ($t \in \mathbb{N}$) either $A_{n+1}(\alpha) \neq 0$ or $B_n(\alpha) \neq 0$. In that case $P_0(\alpha) = A_{n+1}(\alpha)$ and $P_1(\alpha) = -B_n(\alpha)$, then $P_{n-1}(\alpha)$ is a corresponding non-null eigenvector of K_n , we acquire that α is an eigenvalue of the matrix K_n .

Case II. Suppose for $n = 2t - 1$ ($t \in \mathbb{N}$) either $A_n(\alpha) \neq 0$ or $B_{n+1}(\alpha) \neq 0$. In that case $P_0(\alpha) = A_n(\alpha)$ and $P_1(\alpha) = -B_{n+1}(\alpha)$, then $P_{n-1}(\alpha)$ is a corresponding non-null eigenvector of K_n , we acquire that α is an eigenvalue of the matrix K_n .

Case III. Suppose for $n = 2t$ ($t \in \mathbb{N}$), $A_n(\alpha) = B_{n+1}(\alpha) = 0$ and $B_n(\alpha) \neq 0$. Let $F_{n-1}(\alpha) = -A_{n-1}(\alpha)B_n(\alpha)$, we own $K_n F_{n-1}(\alpha) = \alpha F_{n-1}(\alpha)$, then $F_{n-1}(\alpha)$ is a corresponding non-null eigenvector of K_n , we acquire that α is an

eigenvalue of the matrix K_n .

Case IV. Suppose for $n = 2t + 1$ ($t \in \mathbb{N}$), $A_{n-1}(\alpha) = B_n(\alpha) = 0$ and $A_n(\alpha) \neq 0$. Let $F_{n-1}(\alpha) = A_n(\alpha)B_{n-1}(\alpha)$, we own $K_n F_{n-1}(\alpha) = \alpha F_{n-1}(\alpha)$, then $F_{n-1}(\alpha)$ is a corresponding non-null eigenvector of K_n , we acquire that α is an eigenvalue of the matrix K_n . ■

Theorem 6 *Let K_n be n -square pentadiagonal 2-Toeplitz matrix and the corresponding polynomial P_{n+1} in the eq. (10). Suppose that P_{n+1} has simple zeros, the characteristic polynomial of K_n is completely*

$$\begin{aligned} |xI_n - K_n| &= (b_1 b_2)^{\frac{n}{2}} A_{n+1} B_n, & \text{if } n = 2t \ (t \in \mathbb{N}) \\ |xI_n - K_n| &= b_1^{\frac{n+1}{2}} b_2^{\frac{n-1}{2}} A_n B_{n+1}, & \text{if } n = 2t + 1 \ (t \in \mathbb{N}) \end{aligned} \quad (12)$$

here I_n is the n -square identity matrix.

Proof. Let $\alpha_1, \dots, \alpha_n$ the zeros of characteristic polynomial of K_n . Lemma 5, $\alpha_1, \dots, \alpha_n$ are the eigenvalues of the matrix K_n . ■

2 Eigenvalues and eigenvectors of K_n

Theorem 7 *Let K_n be n -square ($n = 2t$, $t \in \mathbb{N}$) pentadiagonal 2-Toeplitz matrix as in (5). Then the eigenvalues and eigenvectors of the matrix K_n are*

$$\alpha_k = \begin{cases} a_1 - 2\sqrt{b_1 c_1} \cos\left(\frac{(k+1)\pi}{n+2}\right), & (k = 2t + 1, t \in \mathbb{N}) \\ a_2 - 2\sqrt{b_2 c_2} \cos\left(\frac{k\pi}{n+2}\right), & (k = 2t, t \in \mathbb{N}) \end{cases} \quad (13)$$

and

$$\begin{bmatrix} B_0(\alpha_j) \\ B_1(\alpha_j) \\ B_2(\alpha_j) \\ \vdots \\ B_{n-2}(\alpha_j) \\ B_{n-1}(\alpha_j) \end{bmatrix} \quad (j = 1, 3, 5, \dots, n-3, n-1); \quad (14)$$

and

$$\begin{bmatrix} A_0(\alpha_j) \\ A_1(\alpha_j) \\ A_2(\alpha_j) \\ \vdots \\ A_{n-2}(\alpha_j) \\ A_{n-1}(\alpha_j) \end{bmatrix} \quad (j = 2, 4, 6, \dots, n-2, n). \quad (15)$$

Proof. We obtain for

$$b_2^{\frac{n}{2}} A_{n+1}(x) = (b_2 c_2)^{\frac{n}{4}} U_{\frac{n}{2}} \left(\frac{x - a_2}{2\sqrt{b_2 c_2}} \right) \quad (16)$$

and

$$b_1^{\frac{n}{2}} B_n(x) = (b_1 c_1)^{\frac{n}{4}} U_{\frac{n}{2}} \left(\frac{x - a_1}{2\sqrt{b_1 c_1}} \right) \quad (17)$$

from the recurrence relations (1) and (2), here $n = 2t$ ($t \in \mathbb{N}$) and $U_n(\cdot)$ is the n th degree Chebyshev polynomial of the second kind [10]:

$$U_n(x) = \frac{\sin((n+1)\arccos x)}{\sin(\arccos x)}$$

All the roots of $U_n(x)$ are included in the interval $[-1, 1]$. Due to (12), (16) and (17), we have

$$\begin{aligned} |xI_n - K_n| &= (b_1 c_1)^{\frac{n}{4}} U_{\frac{n}{2}} \left(\frac{x - a_1}{2\sqrt{b_1 c_1}} \right) (b_2 c_2)^{\frac{n}{4}} U_{\frac{n}{2}} \left(\frac{x - a_2}{2\sqrt{b_2 c_2}} \right) \\ &= (b_1 c_1 b_2 c_2)^{\frac{n}{4}} U_{\frac{n}{2}} \left(\frac{x - a_1}{2\sqrt{b_1 c_1}} \right) U_{\frac{n}{2}} \left(\frac{x - a_2}{2\sqrt{b_2 c_2}} \right). \end{aligned} \quad (18)$$

The eigenvalues of K_n obtained as

$$\alpha_k = \begin{cases} a_1 - 2\sqrt{b_1 c_1} \cos \left(\frac{(k+1)\pi}{n+2} \right), & (k = 2t + 1, t \in \mathbb{N}) \\ a_2 - 2\sqrt{b_2 c_2} \cos \left(\frac{k\pi}{n+2} \right), & (k = 2t, t \in \mathbb{N}) \end{cases}$$

from (18). In [2] Hadj and Elouafi proved eigenvectors of a pentadiagonal matrix. Based on Theorem 6 and [2], we can calculate eigenvectors of the matrix K_n ■

3 The integer powers of the matrix K_n

Considering (14) and (15), we write down the transforming matrices L_n ($n = 2t$, $t \in \mathbb{N}$) as following:

$$L_n = \begin{bmatrix} B_0(\alpha_1) & A_0(\alpha_2) & B_0(\alpha_3) & A_0(\alpha_4) \\ B_1(\alpha_1) & A_1(\alpha_2) & B_1(\alpha_3) & A_1(\alpha_4) \\ B_2(\alpha_1) & A_2(\alpha_2) & B_2(\alpha_3) & A_2(\alpha_4) \\ \vdots & \vdots & \vdots & \vdots \\ B_{n-3}(\alpha_1) & A_{n-3}(\alpha_2) & B_{n-3}(\alpha_3) & A_{n-3}(\alpha_4) \\ B_{n-2}(\alpha_1) & A_{n-2}(\alpha_2) & B_{n-2}(\alpha_3) & A_{n-2}(\alpha_4) \\ B_{n-1}(\alpha_1) & A_{n-1}(\alpha_2) & B_{n-1}(\alpha_3) & A_{n-1}(\alpha_4) \end{bmatrix}$$

$$\begin{bmatrix} \cdots & B_0(\alpha_{n-1}) & A_0(\alpha_n) \\ \cdots & B_1(\alpha_{n-1}) & A_1(\alpha_n) \\ \cdots & B_2(\alpha_{n-1}) & A_2(\alpha_n) \\ \vdots & \vdots & \vdots \\ \cdots & B_{n-3}(\alpha_{n-1}) & A_{n-3}(\alpha_n) \\ \cdots & B_{n-2}(\alpha_{n-1}) & A_{n-2}(\alpha_n) \\ \cdots & B_{n-1}(\alpha_{n-1}) & A_{n-1}(\alpha_n) \end{bmatrix} \quad (19)$$

Now, let us find the inverse matrix L_n^{-1} of the matrix L_n . If we denote i -th row of the inverse matrix L_n^{-1} by μ_i , then we obtain

$$\begin{bmatrix} q_i r_1^l B_0(\alpha_i) \\ q_i r_1^l B_1(\alpha_i) \\ q_i r_1^l B_2(\alpha_i) \\ q_i r_1^l B_3(\alpha_i) \\ q_i r_1^l B_4(\alpha_i) \\ \vdots \\ q_i r_1^l B_{n-4}(\alpha_i) \\ q_i r_1^l B_{n-3}(\alpha_i) \\ q_i r_1^l B_{n-2}(\alpha_i) \\ q_i r_1^l B_{n-1}(\alpha_i) \end{bmatrix}^T \quad (i = 1, 3, 5, \dots, n-3, n-1); \quad (20)$$

and

$$\begin{bmatrix} q_i r_2^l A_0(\alpha_i) \\ q_i r_2^l A_1(\alpha_i) \\ q_i r_2^l A_2(\alpha_i) \\ q_i r_2^l A_3(\alpha_i) \\ q_i r_2^l A_4(\alpha_i) \\ \vdots \\ q_i r_2^l A_{n-4}(\alpha_i) \\ q_i r_2^l A_{n-3}(\alpha_i) \\ q_i r_2^l A_{n-2}(\alpha_i) \\ q_i r_2^l A_{n-1}(\alpha_i) \end{bmatrix}^T \quad (i = 2, 4, 6, \dots, n-2, n) \quad (21)$$

where $r_1 = \sqrt{\frac{b_1}{c_1}}$, $r_2 = \sqrt{\frac{b_2}{c_2}}$,

$$l = \begin{cases} j-1, & j = 1, 3, 5, \dots, n-3, n-1 \\ j-2, & j = 2, 4, 6, \dots, n-2, n \end{cases}$$

and

$$q_i = \begin{cases} \frac{4 - \left(\frac{\alpha_i - a_1}{\sqrt{b_1 c_1}}\right)^2}{n+2}, & i = 2t+1 \ (t \in \mathbb{N}) \\ \frac{4 - \left(\frac{\alpha_i - a_2}{\sqrt{b_2 c_2}}\right)^2}{n+2}, & i = 2t \ (t \in \mathbb{N}). \end{cases}$$

Thus, we obtain

$$L_n^{-1} = \begin{bmatrix} q_1 B_0(\alpha_1) & q_1 B_1(\alpha_1) & q_1 r_1^2 B_2(\alpha_1) & q_1 r_1^2 B_3(\alpha_1) \\ q_2 A_0(\alpha_2) & q_2 A_1(\alpha_2) & q_2 r_2^2 A_2(\alpha_2) & q_2 r_2^2 A_3(\alpha_2) \\ q_3 B_0(\alpha_3) & q_3 B_1(\alpha_3) & q_3 r_1^2 B_2(\alpha_3) & q_1 r_1^2 B_3(\alpha_3) \\ \vdots & \vdots & \vdots & \vdots \\ q_{n-1} B_0(\alpha_{n-1}) & q_{n-1} B_1(\alpha_{n-1}) & q_{n-1} r_1^2 B_2(\alpha_{n-1}) & q_{n-1} r_1^2 B_3(\alpha_{n-1}) \\ q_n A_0(\alpha_n) & q_n A_1(\alpha_n) & q_n r_2^2 A_2(\alpha_n) & q_n r_2^2 A_3(\alpha_n) \\ \cdots & q_1 r_1^{n-2} B_{n-2}(\alpha_1) & q_1 r_1^{n-2} B_{n-1}(\alpha_1) & \\ \cdots & q_2 r_2^{n-2} A_{n-2}(\alpha_2) & q_2 r_2^{n-2} A_{n-1}(\alpha_2) & \\ \cdots & q_3 r_1^{n-2} B_{n-2}(\alpha_3) & q_3 r_1^{n-2} B_{n-1}(\alpha_3) & \\ \ddots & \vdots & \vdots & \\ \cdots & q_{n-1} r_1^{n-2} B_{n-2}(\alpha_{n-1}) & q_{n-1} r_1^{n-2} B_{n-1}(\alpha_{n-1}) & \\ \cdots & q_n r_2^{n-2} A_{n-2}(\alpha_n) & q_n r_2^{n-2} A_{n-1}(\alpha_n) & \end{bmatrix}. \quad (22)$$

We write the sth powers of the matrix K_n as

$$K_n^s = L_n J_n^s L_n^{-1} = W(s) = (w_{ij}(s)). \quad (23)$$

Then

$$w_{ij}(s) = \begin{cases} 0 & , \text{ if } (-1)^{i+j} = -1 \\ \sum_{z=1}^{\frac{n}{2}} q_{2z-1} r_1^l \alpha_{2z-1}^s B_{i-1}(\alpha_{2z-1}) B_{j-1}(\alpha_{2z-1}) & j = 1, 3, \dots, n-1 \\ \sum_{z=1}^{\frac{n}{2}} q_{2z} r_2^l \alpha_{2z}^s A_{i-1}(\alpha_{2z}) A_{j-1}(\alpha_{2z}) & j = 2, 4, \dots, n \end{cases}, \text{ if } (-1)^{i+j} = 1 \quad (24)$$

here $r_1 = \sqrt{\frac{b_1}{c_1}}$, $r_2 = \sqrt{\frac{b_2}{c_2}}$,

$$l = \begin{cases} j-1, & j = 1, 3, 5, \dots, n-3, n-1 \\ j-2, & j = 2, 4, 6, \dots, n-2, n \end{cases},$$

$$q_i = \begin{cases} \frac{4 - \left(\frac{\alpha_i - a_1}{\sqrt{b_1 c_1}}\right)^2}{n+2}, & i = 2t+1 (t \in \mathbb{N}) \\ \frac{4 - \left(\frac{\alpha_i - a_2}{\sqrt{b_2 c_2}}\right)^2}{n+2}, & i = 2t (t \in \mathbb{N}). \end{cases}$$

and α_k are the eigenvalues of the matrix K_n ($n = 2t$, $t \in \mathbb{N}$).

Corollary 8 Let K_n be n -square ($n = 2t$, $t \in \mathbb{N}$; $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{C} \setminus \{0\}$) pentadiagonal 2-Toeplitz matrix as in (5), from Theorem 7

$$a_1 \neq 2\sqrt{b_1 c_1} \cos\left(\frac{(k+1)\pi}{n+2}\right) \quad (25)$$

($k = 2t+1$, $t \in \mathbb{N}$) and

$$a_2 \neq 2\sqrt{b_2 c_2} \cos\left(\frac{k\pi}{n+2}\right). \quad (26)$$

($k = 2t, t \in \mathbb{N}$). In that case, there exists the inverse and negative integer powers of the matrix K_n .

4 Numerical examples

Example 9 Taking $n = 6$ in Theorem 7, we obtain

$$\begin{aligned} J_6 &= \text{diag}(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) \\ &= \text{diag}(a_1 - \sqrt{2b_1c_1}, a_2 - \sqrt{2b_2c_2}, a_1, a_2, a_1 + \sqrt{2b_1c_1}, a_2 + \sqrt{2b_2c_2}) \end{aligned}$$

and

$$\begin{aligned} K_6^s &= L_6 J_6^s L_6^{-1} = W(s) \\ &= (w_{ij}(s)) = \begin{bmatrix} x_1 & 0 & x_7 & 0 & x_{11} & 0 \\ 0 & x_2 & 0 & x_8 & 0 & x_{12} \\ x_3 & 0 & x_9 & 0 & x_7 & 0 \\ 0 & x_4 & 0 & x_{10} & 0 & x_8 \\ x_5 & 0 & x_3 & 0 & x_1 & 0 \\ 0 & x_6 & 0 & x_4 & 0 & x_2 \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} x_1 &= \frac{1}{4} \left[(a_1 + \sqrt{2b_1c_1})^s + 2a_1^s + (a_1 - \sqrt{2b_1c_1})^s \right] \\ x_2 &= \frac{1}{4} \left[(a_2 + \sqrt{2b_2c_2})^s + 2a_2^s + (a_2 - \sqrt{2b_2c_2})^s \right] \\ x_3 &= \frac{\sqrt{2}}{4} r_1^{-1} \left[(a_1 + \sqrt{2b_1c_1})^s - (a_1 - \sqrt{2b_1c_1})^s \right] \\ x_4 &= \frac{\sqrt{2}}{4} r_2^{-1} \left[(a_2 + \sqrt{2b_2c_2})^s - (a_2 - \sqrt{2b_2c_2})^s \right] \\ x_5 &= \frac{1}{4} r_1^{-2} \left[(a_1 + \sqrt{2b_1c_1})^s - 2a_1^s + (a_1 - \sqrt{2b_1c_1})^s \right] \\ x_6 &= \frac{1}{4} r_2^{-2} \left[(a_2 + \sqrt{2b_2c_2})^s - 2a_2^s + (a_2 - \sqrt{2b_2c_2})^s \right] \\ x_7 &= \frac{\sqrt{2}}{4} r_1 \left[(a_1 + \sqrt{2b_1c_1})^s - (a_1 - \sqrt{2b_1c_1})^s \right] \\ x_8 &= \frac{\sqrt{2}}{4} r_2 \left[(a_2 + \sqrt{2b_2c_2})^s - (a_2 - \sqrt{2b_2c_2})^s \right] \\ x_9 &= \frac{1}{2} \left[(a_1 + \sqrt{2b_1c_1})^s + (a_1 - \sqrt{2b_1c_1})^s \right] \\ x_{10} &= \frac{1}{2} \left[(a_2 + \sqrt{2b_2c_2})^s + (a_2 - \sqrt{2b_2c_2})^s \right] \\ x_{11} &= \frac{1}{4} r_1^2 \left[(a_1 + \sqrt{2b_1c_1})^s - 2a_1^s + (a_1 - \sqrt{2b_1c_1})^s \right] \\ x_{12} &= \frac{1}{4} r_2^2 \left[(a_2 + \sqrt{2b_2c_2})^s - 2a_2^s + (a_2 - \sqrt{2b_2c_2})^s \right]. \end{aligned}$$

Example 10 Taking $s = 3, n = 8, a_1 = 1, a_2 = i + 1, b_1 = 3, b_2 = i + 3, c_1 = 5$ and $c_2 = i + 5$ in Theorem 7, we obtain

$$\begin{aligned} J_8 &= \text{diag}(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8) \\ &= \text{diag}(-5.267, -5.280 - 0.668i, -1.394, -1.399 - 0.363i, \\ &\quad 3.3394, 3.399 + 1.637i, 7.267, 7.280 + 2.668i) \end{aligned}$$

and

$$\begin{aligned}
K_8^3 &= L_8 J_8^3 L_8^{-1} = W(3) \\
&= (w_{ij}(3)) = \begin{bmatrix} 46 & 0 & 99 & 0 & 27 \\ 0 & 16 + 68i & 0 & 62 + 94i & 0 \\ 165 & 0 & 91 & 0 & 144 \\ 0 & 118 + 138i & 0 & 34 + 134i & 0 \\ 75 & 0 & 240 & 0 & 91 \\ 0 & 42 + 102i & 0 & 180 + 192i & 0 \\ 125 & 0 & 75 & 0 & 165 \\ 0 & 110 + 74i & 0 & 42 + 102i & 0 \end{bmatrix} \\
&\quad \begin{bmatrix} 0 & 27 & 0 \\ 6 + 42i & 0 & 18 + 26i \\ 0 & 27 & 0 \\ 96 + 12i & 0 & 6 + 42i \\ 0 & 99 & 0 \\ 34 + 134i & 0 & 62 + 94i \\ 0 & 46 & 0 \\ 118 + 138i & 0 & 16 + 68i \end{bmatrix}.
\end{aligned}$$

Example 11 Taking $s = -4, n = 10, a_1 = 1, a_2 = 2, b_1 = 3, b_2 = 4, c_1 = 5$ and $c_2 = 6$ in Theorem 7, we get

$$\begin{aligned}
J_{10} &= \text{diag}(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, \alpha_9, \alpha_{10}) \\
&= \text{diag}(-5.7082, -6.4853, -2.8730, -2.8990, 1, 2, 4.8730, 6.8990, 7.7082, 10.4853)
\end{aligned}$$

and

$$\begin{aligned}
K_{10}^{-4} &= L_{10} J_{10}^{-4} L_{10}^{-1} = W(-4) \\
&= (w_{ij}(-4)) = \begin{bmatrix} 0.3375 & 0 & -0.0026 & 0 & -0.1999 \\ 0 & 0.0245 & 0 & -0.0029 & 0 \\ -0.0043 & 0 & 0.0044 & 0 & -0.0001 \\ 0 & -0.0043 & 0 & 0.0038 & 0 \\ -0.5552 & 0 & -0.0002 & 0 & 0.3337 \\ 0 & -0.0311 & 0 & -0.0002 & 0 \\ 0.0067 & 0 & -0.0063 & 0 & -0.0002 \\ 0 & 0.0062 & 0 & -0.0052 & 0 \\ 0.9148 & 0 & 0.0067 & 0 & -0.5552 \\ 0 & 0.0388 & 0 & 0.0062 & 0 \end{bmatrix}
\end{aligned}$$

$$\begin{bmatrix} 0 & 0.0015 & 0 & 0.1186 & 0 \\ -0.0138 & 0 & 0.0018 & 0 & 0.0077 \\ 0 & -0.0023 & 0 & 0.0015 & 0 \\ -0.0001 & 0 & 0.0023 & 0 & 0.0018 \\ 0 & -0.0001 & 0 & -0.1999 & 0 \\ 0.0210 & 0 & -0.0001 & 0 & -0.0138 \\ 0 & 0.0044 & 0 & -0.0026 & 0 \\ -0.0001 & 0 & 0.0038 & 0 & -0.0029 \\ 0 & -0.0043 & 0 & 0.3375 & 0 \\ 0.0311 & 0 & -0.0043 & 0 & 0.0245 \end{bmatrix}.$$

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